Fourier Analysis 04-23

Review.

Thm (Uniqueness)

Let $u(x,y) \in C^2(\mathbb{R}\times\mathbb{R}_+) \cap C(\overline{\mathbb{R}\times\mathbb{R}_+})$.

Suppose $\begin{cases} \Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \text{ on } \mathbb{R}\times\mathbb{R}_+. \\ u(x,0) = 0 \end{cases}$

Moreover, suppose $u(x,y) \to 0$ as $|x| + y \to +\infty$

Then $u(x,y) \equiv 0$ on $\mathbb{R}\times\mathbb{R}_+$.

Remark: The assumption $u \to 0$ "at infinity" can not be dropped.

For example: If letting $u(x,y) = y$, ...

Lemma (Mean value property of harmonic functions)

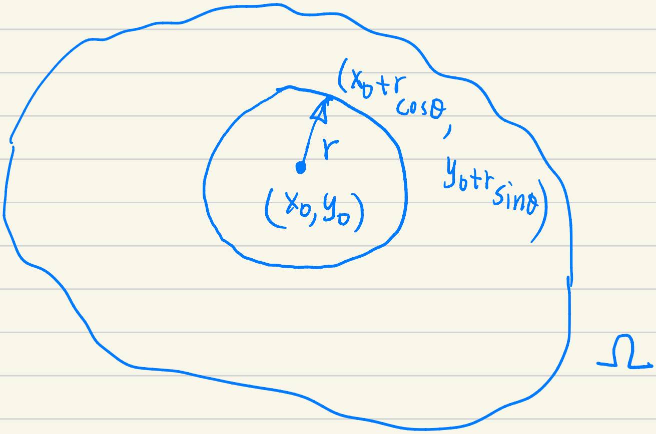
Let Ω be an open set in $\mathbb{R}^2$. Let $u \in C^2(\Omega)$. Suppose $B_R(x_0, y_0) \subseteq \Omega$

$\Delta u = 0$

where $B_R(x_0,y_0) := \{ (x,y) \in \mathbb{R}^2 : (x-x_0)^2 + (y-y_0)^2 \leq R^2 \}$

Then $\forall \ 0 < r < R$,

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r\cos\theta, \ y_0 + r\sin\theta) \, d\theta$$



Maximum principle for harmonic functions:

- Let $\Omega \subset \mathbb{R}^2$ be open and connected. Suppose $u \in C^2(\Omega)$ and $\Delta u = 0$.

  If furthermore u takes the maximum (or minimum) at some point $(x_0, y_0) \in \Omega$. Then u is constant on Ω.

Proof. Let $\gamma = u(x_0, y_0) = \sup\limits_{(x,y)\in\Omega} u(x,y)$.

Set $\tilde{\Omega} = \{(x,y) \in \Omega : \ u(x,y) = \gamma\}$.

Then $(x_0, y_0) \in \tilde{\Omega}$, so $\tilde{\Omega} \neq \emptyset$.

As $u \in C(\Omega)$, $\tilde{\Omega}$ is a relatively closed in Ω, that is, if $(x_n, y_n) \in \tilde{\Omega}$ converges to some $(x,y) \in \Omega$, then $(x,y) \in \tilde{\Omega}$. Hence, $\Omega \setminus \tilde{\Omega} = \{(x,y) \in \Omega : u(x,y) < \gamma\}$ is open.

Next we prove that $\tilde{\Omega}$ is open. To this end, let $(x_1, y_1) \in \tilde{\Omega}$. Choose $r > 0$ such that

$$B_r(x_1, y_1) \subset \Omega.$$

By the mean value property, for any $0 \leq r' < r$,

$$\frac{1}{2\pi} \int_0^{2\pi} u(x_1 + r'\cos\theta,\ y_1 + r'\sin\theta)\, d\theta = \gamma.$$

By the maximality of γ, we have

$$u(x_1 + r'\cos\theta,\ y_1 + r'\sin\theta) = \gamma$$

for all $0 \leq r' < r$, $0 \leq \theta < 2\pi$. Therefore

$$\{(x, y) : (x - x_1)^2 + (y - y_1)^2 < r^2\} \subset \tilde{\Omega},$$

so $\tilde{\Omega}$ is open.

As Ω is connected, both $\tilde{\Omega}$ and $\Omega \setminus \tilde{\Omega}$ are open, we have $\tilde{\Omega} = \Omega$.

$\blacksquare$

<u>Corollary</u>: Let Ω be a bounded open region on $\mathbb{R}^2$. Suppose

$$u \in C^2(\Omega) \cap C(\bar{\Omega}) \quad \text{and} \quad \Delta u = 0$$

Moreover suppose $u = 0$ on $\partial\Omega$

Then $u \equiv 0$ on Ω.

Proof. Since $\bar{\Omega}$ is compact, so u take $\begin{cases} \text{maximum value} \\ \text{minimum value} \end{cases}$ in $\bar{\Omega}$. If both these two values are taken on $\partial\Omega$, then $u \equiv 0$ on $\bar{\Omega}$.

Otherwise, if one of them is taken in Ω, then u is constant on $\bar{\Omega}$ $\Rightarrow$ $u \equiv 0$ on $\bar{\Omega}$.

Poisson summation formula:

Let $f \in \mathcal{M}(\mathbb{R})$ and suppose also that $\hat{f} \in \mathcal{M}(\mathbb{R})$.

Set $F(x) = \sum_{n\in\mathbb{Z}} f(x+n)$. Then

F is a continuous 1-perodic function on $\mathbb{R}$.

Thm 1 (Poisson summation formula)

$$F(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{2\pi i n x}, \quad \forall\, x \in \mathbb{R}.$$

In particular

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

Pf. We first calculate the Fourier coefficients of F on the unit circle.

$$\hat{F}(n) = \int_0^1 F(x)\, e^{-2\pi i n x}\, dx$$

$$= \int_0^1 \sum_{m\in\mathbb{Z}} f(x+m)\, e^{-2\pi i n x}\, dx$$

$$= \sum_{m\in\mathbb{Z}} \int_0^1 f(x+m)\, e^{-2\pi i n x}\, dx$$

$$= \sum_{m\in\mathbb{Z}} \int_m^{m+1} f(y)\, e^{-2\pi i n (y-m)}\, dy$$

$$= \sum_{m\in\mathbb{Z}} \int_m^{m+1} f(y)\, e^{-2\pi i n y}\, dy$$

$$= \int_{-\infty}^{\infty} f(y)\, e^{-2\pi i n y}\, dy$$

$$= \hat{f}(n).$$

Since $\hat{f} \in M(\mathbb{R})$, $\sum |\hat{f}(n)| < \infty$,

So

$$F(x) = \sum_{n\in\mathbb{Z}} \hat{F}(n)\, e^{2\pi i n x}.$$

$$= \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{2\pi i n x}$$

$\blacksquare$

Example 1.:

Let $f(x) = \frac{1}{\pi} \frac{y}{x^2+y^2}$ $(y>0)$

$(f(x) \equiv P_y(x))$.

$$\hat{f}(\xi) = e^{-2\pi|\xi|\cdot y}$$

By Poisson summation formula, we have

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{2\pi i n x}$$

i.e.

$$\frac{1}{\pi} \sum_{n\in\mathbb{Z}} \frac{y}{(x+n)^2+y^2} = \sum_{n\in\mathbb{Z}} e^{-2\pi|n|y} e^{2\pi i n x}$$

Letting $x=0$ gives

$$\frac{1}{\pi} \sum_{n\in\mathbb{Z}} \frac{y}{n^2+y^2} = \sum_{n\in\mathbb{Z}} e^{-2\pi|n|y}$$

$$= 1 + 2 \sum_{n=1}^{\infty} e^{-2\pi|n|y}$$

$$= 1 + 2 \cdot \frac{e^{-2\pi y}}{1 - e^{-2\pi y}}$$

$$= \frac{1 + e^{-2\pi y}}{1 - e^{-2\pi y}}$$

Hence

$$\sum_{n \in \mathbb{Z}} \frac{1}{n^2 + y^2} = \frac{\pi}{y} \cdot \frac{1 + e^{-2\pi y}}{1 - e^{-2\pi y}} .$$

Lettig $y = 1$ gives

$$\sum_{n \in \mathbb{Z}} \frac{1}{n^2 + 1} = \pi \cdot \frac{1 + e^{-2\pi}}{1 - e^{-2\pi}} .$$

<u>Example 2</u>: Consider the Theta function

$$\Theta(s) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 s}, \qquad s>0$$

It satisfies the following property

$$s^{-\frac{1}{2}} \cdot \Theta\left(\frac{1}{s}\right) = \Theta(s), \qquad s>0.$$

To see this, let

$$f(x) = e^{-\pi x^2 \cdot s} \qquad (s>0)$$

$$\left(e^{-\pi x^2} \xmapsto{\mathcal{F}} e^{-\pi \xi^2} \right)$$

$$f(x) \xrightarrow{\mathcal{F}} \frac{1}{\sqrt{s}} \cdot e^{-\pi \left(\frac{\xi}{\sqrt{s}}\right)^2}$$

$$= \frac{1}{\sqrt{s}} e^{-\pi \xi^2/s}.$$

By Poisson Summation formula

$$\sum f(n) = \sum \hat{f}(n)$$

i.e. $$\sum_n e^{-\pi n^2 s} = \sum_n \frac{1}{\sqrt{s}} e^{-\pi n^2/s}$$

So $$\Theta(s) = \frac{1}{\sqrt{s}} \Theta\left(\frac{1}{s}\right).$$